Numerical Ranges and their Convexity

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Numerical Range

Let $A \in \mathbb{M}_n(\mathbb{C})$. The numerical range of A is defined to be the set W(A), where

$$W(A) := \{< x, Ax >: x \in \mathbb{C}^n \ and \ ||x|| = 1\}$$

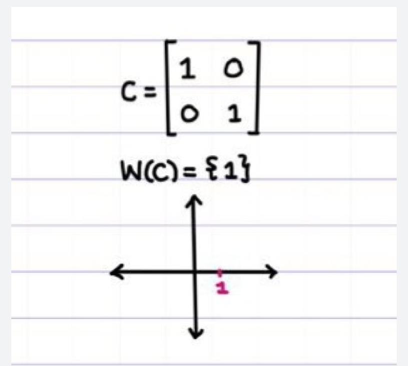

Figure 1. A point

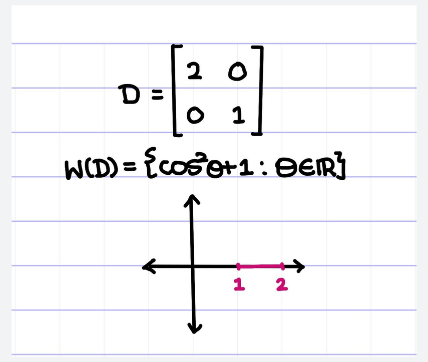

Figure 2. Line segment

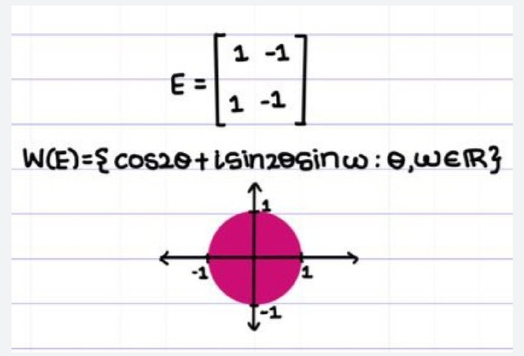

Figure 3. Elliptical

Examples of Numerical Ranges of different Operators

Properties of Numerical Range

Let A,I_n $\in \mathbb{M}_n(\mathbb{C})$ where I_n is the identity matrix, $U \in \mathbb{U}(n)$, $\alpha, \beta, \lambda \in \mathbb{C}$.

1. $W(U^*AU) = W(A)$
2. $W(\alpha A + \beta I_n) = \alpha W(A) + \beta$
3. If λ is an eigenvalue of A, then $\lambda \in W(A)$
4. If A is normal, then $W(A) = co(\sigma(A))$
5. W(A) is closed

Toeplitz-Hausdorff Theorem:

For all A$\in \mathbb{M}_n(\mathbb{C})$, W(A) is convex.

Generalizations of Numerical Range

Let A, C $\in \mathbb{M}_n(\mathbb{C})$, c= $(c_1, ..., c_n) \in \mathbb{C}^n$, and $1 \leq k \leq n$.

k-Numerical Range

The k-Numerical range of A is defined to be the set W_k(A), where

$$W_k(A) := \{\sum_{i=1}^{k} < x_i, Ax_i >: x_i \in \mathbb{C}^n \ and \ x_i's \ are \ orthonormal\}$$

Halmos-Berger Theorem:

For all A$\in \mathbb{M}_n(\mathbb{C})$, W_k(A) is convex.

c-Numerical Range

The c-Numerical Range of A is described as the set W_c(A), where

$$W_c(A) := \{\sum_{i=1}^{k} ci < x_i, Ax_i >: x_i \in \mathbb{C}^n \ and \ x_i's \ form \ an \ orthonormal \ basis\}$$

Theorem:

For all A$\in \mathbb{M}_n(\mathbb{C})$ and c $\in \mathbb{R}^n$, W_c(A) is convex .

Special Case: For all A$\in \mathbb{M}_2(\mathbb{C})$ and c $\in \mathbb{C}^2$, W_c(A) is convex .

C-Numerical Range

The C-Numerical Range of A is described to be the set W_C(A), where

$$W_C(A) = \{tr(CUAU^*) : U \in U(n)\}.$$

Theorem:

For all A,C$\in$M$_n(\mathbb{C})$, such that C is hermitian, W_C(A) is convex.

Joint Numerical Range

Joint Numerical Range

The Joint Numerical Range of $B = (B_1, ..., B_m) \in (\mathbb{M}_n(\mathbb{C}))^m$ is described to be the set W(B), where

$$W(B) := \{(x^*B_1x, ..., x^*B_mx) : \ x \in \mathbb{C}^n \ and \ x^*x = 1\}$$

Joint r-Numerical Range

Let $1 \leq r \leq n-1$. The r-Joint Numerical Range of $B = (B_1, ..., B_m)$ is described to be the set W$^{(r)}$(B), where

$$W(B) := \{(\sum_{i=1}^{r} x_i^*B_1x_i, ..., \sum_{i=1}^{r} x_i^*B_mx_i) : \ x_i \in \mathbb{C}^n \ and \ \sum_{i=1}^{r} x_i^*x_i = 1\}$$

Properties of Joint Numerical Range

For all $A_1, ..., A_m, U \in \mathbb{M}_n(\mathbb{C})$, such that U is unitary, $F = \{A_1, ..., A_m\}$ and $B = \{B_1, ..., B_k\}$ is a basis of span(F) and for all $i \in \{1, ..., m\}$, $A_i = H_i + iG_i$ where H_i and G_i are Hermitian matrices, then the following are true:

1. $W(A_1, ..., A_m) = W(U^*A_1U, ..., U^*A_mU)$
2. $W(A_1, ..., A_m) = W(A_1^t, ..., A_m^t)$
3. $W(A_1, ..., A_m)$ is convex iff $W(B_1, ..., B_k)$ is convex.
4. If $W(A_1, ..., A_m)$ is convex then so is $W(C_1, ..., C_s)$ for any family C = $\{C_1, ..., C_s\} \subset$ Span(F).
5. The family F is commuting iff the basis B is commuting.
6. $W(A_1, ..., A_m) \subset \mathbb{C}^m$ can be identified with $W(H_1, G_1..., H_m, G_m) \subset \mathbb{R}^{2m}$

References

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Convexity of Joint Numerical Range:

1. For all $A_1, A_2 \in \mathbb{H}_n$, $W(A_1, A_2)$ is convex.
2. For $n = 2$, and $H_1, ..., H_m \in \mathbb{H}_2$, $W(H_1, ..., H_m)$ is convex iff Span$\{I_2, H_1, ..., H_m\} \neq \mathbb{H}_2$
3. For $n \geq 3$ and $H_1, ..., H_m \in \mathbb{H}_n$, if dim(Span{I$_n$,H$_1$,...,H$_m$}) ≤ 4, then W(H$_1$,...,H$_m$) is convex.
4. For any commuting family $F = \{A_1, ..., A_m\} \subset \mathbb{M}_2(\mathbb{C})$, $W(A_1, ..., A_m)$ is convex.
5. There exists commuting matrices $A_1, A_2, A_3 \in \mathbb{M}_3(\mathbb{C})$ such that $W(A_1, A_2, A_3)$ is not convex. Extending this, we get that for all $n \geq 3$, there exists commuting matrices $A_1, A_2, A_3 \in \mathbb{M}_n(\mathbb{C})$ such that $W(A_1, A_2, A_3)$ is not convex.

Result:

Let $B_1, ..., B_r \in \mathbb{H}_n$ be Hermitian Matrices. For every unit vector $\mathbf{v} = (v_1, ..., v_r) \in \mathbb{R}^r$, let $P_v = \{b \in \mathbb{R}^r : b * v \leq \lambda_1(v_1B_1 + ... + v_rB_r)\}$, where $\lambda_1(H)$ denotes the largest eigenvalue of $H \in \mathbb{H}_n$ and $b * v = \Sigma_{i=1}^r b_iv_i$ for $b = (b_1, ..., b_r) \in \mathbb{R}^r$. Then, Conv$W(B_1, ..., B_r)$ = $\bigcap\{P_v : v = (v_1, ..., v_r) \in \mathbb{R}^r, v * v = 1\}$. Consequently, $\partial P_v \cap W(B_1, ..., B_r)$ = $\{(x*, B_1x, ..., x * B_rx) : x \in \mathbb{C}^n, x * x = 1, B_vx = \lambda_1(B_v)x\}$ where $B_v = \Sigma_{i=1}^r v_iB_i$.

Convexity of Joint r-Numerical Range

Let $f_{\mathbb{F}}(n)$ be the dimension of the real linear space $\mathbb{H}_n(\mathbb{F})$. Then the following theorems are equivalent.

Theorem 1:

Let $1 \leq r \leq n-1$. If p< $f_{\mathbb{F}}(r+1) - \delta_{n,r+1}$, then for any $A_1, ..., A_p \in \mathbb{H}_n(\mathbb{F})$, $W^{(r)}(A_1, ..., A_p)$ is convex where $\delta_{i,j}$ is the kronecker delta.

Theorem 2: *(Bohnenblust)*

Let $1 \leq r \leq n-1$. If p< $f_{\mathbb{F}}(r+1) - \delta_{n,r+1}$, then for any $A_1, ..., A_p \in \mathbb{H}_n(\mathbb{F})$ such that $(\sum_{i=1}^r x_i^*A_1x_i, ..., \sum_{i=1}^r x_i^*A_px_i) \neq (0, ..., 0)$ for all $x \in (\mathbb{F}^n)^p \setminus \{(0, 0, ..., 0)\}$, then there exists $a_1, ..., a_p \in \mathbb{R}$ such that $\sum_{i=1}^p a_iA_i > 0$.

Joint Numerical Range of three Hermitian Matrices

Let $A_1, A_2, A_3 \in \mathbb{H}_n(\mathbb{C})$. Then $W(A_1, A_2, A_3)$ has a convex boundary. Moreover, if n>2 then $W(A_1, A_2, A_3)$ is convex. However it is not true for n=2 since, $W(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}) = S^2$, the unit sphere which in not convex.

Uses of Joint Numerical Range

1. Used to study the properties of commutators and compressions of operators.
2. Help determining minimal Hermitian matrices, which are important in various optimization problems.
3. Used to study the properties of the moment of a subspace.

Prospective Work

1. Investigate the properties of the Joint Numerical Range of normal matrices.
2. Explore the properties of the Joint Numerical Range of unitary matrices.
3. Examine whether the results from (1) and (2) can be unified to generalize for any tuple of matrices using Singular Value Decomposition (SVD).